Indian Statistical Institute, Bangalore
B. Math. First Year
First Semester - Analysis I

Semestral Exam Date : Nov 05, 2014

This paper carries 44 marks. Maximum marks you can get is 34

1. Let $x_1 \geq x_2 \geq ..... \geq 0$. Show that $\sum_1^\infty x_j$ is finite if and only if $\sum_0^\infty y_j < \infty$ where $y_k = 2^k x_{2^k}$ for $k = 0, 1, 2, 3....$. [4]
2. Let $f : [a, b] \to \mathbb{R}$ be continuous and twice differentiable. Show that there exists c in (a, b) such that $f(b) = f(a) + (b - a)f'(a) + \frac{(b-a)^2}{2!} f''(c)$ [5]
3. Let $y_1, y_2, y_3...$ be any Cauchy sequence of reals. Without using the completeness of $\mathbb{R}$, show that the sequence $y_1, y_2...$ is a bounded sequence. [2]
4. Show that the complex numbers $\mathbb{C}$ is complete. [You can use $\mathbb{R}$ is complete] [3]
5. Let $\{x_n\}_{n=1}^\infty$ be a sequence defined by $x_1 = \frac{1}{2}$ and, for any $n \geq 1$,
$$x_{n+1} = \frac{x_n^2}{x_n^2 - x_n + 1}$$
prove that $\sum_{n=1}^\infty x_n$ is convergent. [5]
6. (a) Let u_n be a sequence of complex numbers with $\sum |u_n| < \infty$. Show that $\sum_1^\infty u_n^2$ exists. [3]
   (b) Give an example $a_1, a_2, ...$ a sequence of real numbers such that $\sum_1^\infty a_n$ exists but $\sum a_n^2 = \infty$ and prove your claim. [2]
7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function satisfying
$$f(x + y) = f(x) + f(y)$$
for all x, y in $\mathbb{R}$. If f is continuous at x_0, show that f is continuous on the whole of $\mathbb{R}$. [3]
8. (a) Let $f : [0, 1] \to [0, 1]$ be continuous. By considering the function $g(x) = f(x) - x$ or otherwise show that there exists x_0 with $f(x_0) = x_0$ [1]
   (b) Let f be as above and satisfying $f(f(y)) = f(y)$ for all y. Let $E_f = \{x : f(x) = x\}$. If E_f has at least two points then show that it must be an interval. [3]
9. Let $f : [0, 1] \to \mathbb{R}$ be continuous in $[0, 1]$ and differentiable in$(0, 1)$ such that $f(0) = 0$ and $0 \leq f'(x) \leq 2f(x)$, for all $x \in (0, 1)$. Prove that $f(x) = 0$ for all $x \in [0, 1]$. [Hint: $g(x) = e^{-2x} f(x)$ may be useful.] [3]
10. Show that if f is continuous on $[0, \infty)$ and uniformly continuous on $[a, \infty)$ for some positive constant a, then f is uniformly continuous on $[0, \infty)$. [4]
11. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that there is no $x \in [0, 1]$ such that $f(x) = f'(x) = 0$. Show that the set $Z := \{x \in [0, 1] : f(x) = 0\}$ is finite. [3]
12. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(r + \frac{1}{n}) = f(r)$ for any rational number r and positive integer n. Prove that f is constant. [Hint: Is $f(r - \frac{1}{n}) = f(r)$ also for rational r and $n = 1, 2, 3...$] [3]